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Large-N quantum gauge theories in two dimensions

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Abstract.

The partition function of a two-dimensional quantum gauge theory in the large-N limit is expressed as the functional integral over some scalar field. The large-N saddle point equation is presented and solved. The free energy is calculated as the function of the area and of the Euler characteristic. There is no non-trivial saddle point at genus $g > 0$. The existence of a non-trivial saddle point is closely related to the weak coupling behavior of the theory. Possible applications of the method to higher dimensions are briefly discussed.

It is well known that quantum gauge theories are exactly soluble in two dimensions. It makes two dimensional model a useful instrument for probing a new methods eventually intended to higher dimensions. In this letter we develop a new large-N approach for the gauge theories, and apply it to the two-dimensional pure gauge theory. We consider the model in the lattice formulation, given by K.Wilson [1]. A convenient approach to this model at arbitrary N is the group-theoretical expansion proposed by A.A.Migdal [2]. This approach has been applied by the author to calculate loop averages[1] and the partition function of the model in two dimensions at arbitrary finite N [5]. The results were expressed through sums over irreducible representations of the gauge group.

The idea we develop below is that in the large-N limit the finite-N signatures (parametrized by the N numbers), can be replaced by a scalar field, while the sum over all signatures can be represented by the functional integral over this scalar field.

Using the exact results of [5], we realize this program in two dimensions (we consider the case of arbitrary *orientable* surfaces). We solve the large-N saddle point equation by the method described in ref.[6] and calculate the free energy as a function of the area and of the topological characteristics of the manifold.

In conclusion we discuss the difference between our approach and that of ref.[7] and the possibility of applying both of them together to higher-dimensional models.

Thus, we consider the lattice model [1]

$$S = \beta_0 N \sum_f \mathrm{tr}\,[\,U_f + U_f^\dagger], \tag{1}$$

where sum goes over all faces (fundamental polygons) f of the two-dimensional lattice and β_0 is the lattice coupling constant.

Following [2] we expand the contribution of each face over irreducible representations, r, of the (compact) gauge group :

$$\mathrm{e}^{\,\beta_0 N\,\mathrm{tr}\,[\,U_f+U_f^\dagger]} = \sum_r \mathrm{d}_{\,r}\lambda_r(\beta_0 N)\chi_r(U_f), \tag{2}$$

where $\chi_r(U)$ and $\mathrm{d}_{\,r} = \chi_r(I)$ are characters and dimensions of r's respectively. The coefficients of the expansion (2)

$$\lambda_r(\beta_0 N) = \int DU\,\mathrm{e}^{\,\beta_0 N\,\mathrm{tr}\,[\,U+U^\dagger]}\chi_r(U) \tag{3}$$

[1]All loop averages on a *plane* were calculated by V.A.Kazakov and I.K.Kostov [3]. See also paper of N.Bralic [4].

have the following asymptotic behavior in the continuum limit $\epsilon \to 0$ (ϵ is the area of the face and $\beta = \epsilon\beta_0$) :

$$\lambda_r(\beta N) \;\sim\; \exp\left(-\frac{C_2(r)}{2\beta N}\epsilon\right), \tag{4}$$

where $C_2(r)$ is the eigenvalue of the quadratic Casimir operator.

In ref.[5] the following expression for the partition function has been derived[2]:

$$Z = \sum_r \mathrm{d}_r^{\,\eta} \exp\left(-\frac{C_2(r)A}{2\beta N}\right) \tag{5}$$

where A is the area of the surface and $\eta = 2 - 2g$ is the Euler characteristic, with g being a number of the handles (genus). It has been argued in [5] that $Z = 1$ in the case of the surface with holes.

Now we substitute in eq.(5) an explicit expressions for dimensions,

$$\mathrm{d}_r = \prod_{i<j}^{N}\left(1 + \frac{n_i - n_j}{j - i}\right), \tag{6}$$

and Casimir eigenvalues,

$$C_2(r) = \sum_{k=1}^{N}\left(n_k^2 + n_k(N - 2k + 1)\right), \tag{7}$$

where n_k's are parameters of signature: $r = \{n_1, \ldots, n_N\}$ obeying the dominance condition $n_1 \geq \ldots \geq n_N$. Hence, eq.(5) reads

$$Z = \sum_{n_1}^{\infty}\ldots\sum_{n_N}^{\infty}\prod_{k=1}^{N-1}\theta(n_k - n_{k+1})\,\mathrm{e}^{\,S}, \tag{8}$$

$$S = -\frac{A}{2\beta N}\sum_{k=1}^{N}\left(n_k^2 + n_k(N-2k+1)\right) + \frac{\eta}{2}\sum_{i\neq j}\log\left(1 + \frac{n_i - n_j}{j - i}\right), \tag{9}$$

where the step function, $\theta(n) = 1$ if $n \geq 0$ and $\theta(n) = 0$ if $n < 0$, realizes the dominance condition for signatures.

For large N, we introduce a continuum time, $0 \leq x = \frac{k}{N} \leq 1$, and replace sum over n_k's by the path integral over the scalar field $n(x)$:

$$Z = \int \prod_{1\leq x\leq 0} dn(x)\;\; \mathrm{e}^{\,S} \tag{10}$$

$$S = \frac{N^2}{2}\int_0^1\!dx\left\{-\frac{A}{\beta}\left(n^2(x) + n(x)(1-2x)\right) + \eta\, ⨍_0^1\!dy\log\left(1 + \frac{n(x)-n(y)}{y-x}\right)\right\}. \tag{11}$$

[2] In ref.[5] this formula was obtained actually for $U(N)$ and $SU(N)$ gauge groups but, as it can be easily proven, it holds for any compact gauge group.

We omit here the step function since its contribution to the action is of order N (i.e., $1/N$ with respect to expression (11)) [3].

Now, we calculate (10) using the saddle point method. First, we replace $n(x)$ by the new field $\phi(x) = n(x) - x + \frac{1}{2}$. Then, the saddle point equation is

$$2\xi\phi(x) = \fint_0^1 \frac{dy}{\phi(x) - \phi(y)} \quad ; \quad \xi = \frac{A}{\beta\eta} \quad . \tag{12}$$

Introducing the density

$$\rho(\phi) = \frac{dx}{d\phi} \tag{13}$$

which should be positive, even and normalized to

$$\int_{-a}^{a} d\lambda \, \rho(\lambda) = 1, \tag{14}$$

we rewrite eq.(12) as equation for ρ:

$$2\xi\lambda = \fint_{-a}^{a} \frac{d\mu\rho(\mu)}{\lambda - \mu} \quad ; \quad |\lambda| \leq a \, . \tag{15}$$

The solution of eq.(15) is [4]

$$\rho(\lambda) = \frac{2}{\pi}\sqrt{\xi(1 - \xi\lambda^2)} \tag{16}$$

with

$$a = \frac{1}{\sqrt{\xi}}. \tag{17}$$

Now, we transform (11) into

$$S = \frac{N^2}{2}\left\{\frac{A}{12\beta} + \frac{3}{2}\eta - \frac{A}{\beta}\int_{-a}^{a} d\lambda \, \rho(\lambda)\left(\lambda^2 - \xi^{-1}\fint_{-a}^{a} d\mu \, \rho(\mu)\log|\lambda - \mu|\right)\right\}. \tag{18}$$

Then, integrating (15) with respect to λ and defining the free energy as $F = \frac{2}{N^2}\log Z$ we have

$$F = \frac{A}{12\beta} + \frac{3}{2}\eta + \eta\int_{-a}^{a} d\lambda \, \rho(\lambda)\log|\lambda| \tag{19}$$

and, finally,

$$F = \frac{A}{12\beta} + \eta\left(1 + \frac{1}{2}\log\frac{\eta\beta}{4A}\right) \, . \tag{20}$$

[3] More detailed discussion of this point will be given in [8].

[4] The method of solution of such an equations can be found in ref.[6].

We see from (16),(17) that there is no non-trivial large-N saddle point for genus $g > 0$ (ξ and η become negative). The appearance of this phenomenon is clear already from formula (5). A saddle point exists only when the topological (entropy) term in the effective action (this term arises from powers of d_r's) acts in the opposite direction to the area (energy) term.

In the case of a torus ($g = 1$) there is no topological term. The corresponding free energy is $F = \frac{A}{12\beta}$.

At higher genera, $\eta < 0$, there are only negative powers of d_r in (5) and the topological term acts in the same direction as the area term. The partition function (5) in the large-N limit is then dominated by the trivial representation ($\mathrm{d}_r = 1,\ C_2(r) = 0$) and corresponding free energy equal to zero.

The non-trivial large-N behavior occurs only for a sphere ($g = 0$). In this case, the positive powers of d_r's give a positive contribution to the effective action and compensate (at the saddle point) the negative contribution of the area term.

To summarize, we write the free energy of the large-N quantum gauge theory on an orientable surface with g handles and with h holes,

$$F = \begin{cases} \frac{A}{12\beta} + 2 + \log\frac{\beta}{2A} & , \quad \text{sphere} : g = 0 \text{ and } h = 0 \\ \\ \frac{A}{12\beta} & , \quad \text{torus} : g = 1 \text{ and } h = 0 \\ \\ 0 & , \quad g > 1 \text{ or (and) } h > 0 \end{cases} \tag{21}$$

This formula sums all planar diagrams of the quantum gauge theory in two dimensions.

Note, that the non-trivial large-N saddle point exists only when in the weak coupling limit ($\beta \to \infty$ in our notation) the free energy is divergent. Apparently, this is the case in higher dimensions and we can hope that our method will be relevant there.

Possible application of our approach to higher dimensions is intimately connected to the problem of the (one-link) integration over unitary matrix which is already not so simple as in two dimensions, even at large N. This problem has been studied by D.Gross and E.Witten in [7] and a third order phase transition was found. The technical reason for such a phase transition is the unitarity constraint: the eigenvalue density (the analogue of our quantity (13)),

$$\rho(\alpha) = \frac{2}{\pi}\beta\cos(\alpha)\sqrt{\frac{1}{2\beta} - \sin^2(\alpha)} \tag{22}$$

(where $\alpha(x)$ is the scalar field coming from the unitary matrix eigenvalues) depends on the functions bounded by 1, and, consequently, there are two different types of behavior for coupling constant $\beta > \frac{1}{2}$ and for $\beta < \frac{1}{2}$. From the point of view of our approach this phenomenon seems a lattice artifact, since considering the model, where integration over unitary matrices (over α's at large N) is performed from very beginning [5], we do not realize a phase transition with respect to β.

To conclude, the following proposal for higher dimensions can be made. The large-N limit could be described in terms of both a scalar field $n(x)$ arising from the signatures and a scalar field $\alpha(x)$ arising from eigenvalues of the unitary matrix, with a properly defined integration over α's. Introducing two scalar fields may seems like an unnecessary complication, but this complication could give more freedom to solve the problem.

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